

# Effective Action for a Statistical System with a Field Dependent Wave Function

Pierre Gosselin,<sup>1</sup> Hervé Mohrbach,<sup>2</sup> and Alain Bérard<sup>3,4</sup>

Received November 9, 2004; accepted February 3, 2005

---

We compute the first order correction in  $\hbar$  to the field dependent wave function in Statistical Field Theory. These corrections are evaluated by several usual methods. We limit ourselves to a one dimensional model in order to avoid the usual difficulties with the UV divergences that are not relevant for our purposes. The main result of the paper is that the various methods yield different corrections to the wave function. Moreover, we give arguments to show that the perturbative integration of the exact renormalization group provides the right result.

---

**KEY WORDS:** quantum mechanics; statistical field system; renormalization.

## 1. INTRODUCTION

Recently the first order quantum correction in  $\hbar$  (one-loop) to the classical action of a quantum particle in one dimension, has been computed by Cametti *et al.* (1999). These corrections evaluated to the second order in the derivative expansion of the effective action, using the Iliopoulos *et al.* (1975) expansion, lead to a renormalization of both the potential and the kinetic energy. Borelli and Kleinert (2000) have extended this calculation to a theory with a field dependent wave function. Instead of using the expansion of Iliopoulos *et al.* (1975), they used a method of Frazer (1985) which can be generalized to terms of higher order in derivatives. Note that usually a quantum mechanical system can be considered as a statistical system in one dimension. However, this identification is no more correct for an Hamiltonian with a position dependent mass term. Actually, in quantum mechanics one has to ask for an additional reparametrization invariance (Kleinert and Chervyakov, 2002). In this paper we will consider our system as a

<sup>1</sup> Université Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, UFR de Mathématiques, BP74, 38402 Saint Martin d'Hères, Cedex, France.

<sup>2</sup> Institut Charles Sadron, CNRS UPR 022, 6 rue Boussingault, 67083 Strasbourg Cedex, France.

<sup>3</sup> L.P.M.C.I. Institut de Physique, 1 blvd D.Arago, F-57070, Metz, France.

<sup>4</sup> To whom correspondence should be addressed at L.P.M.C.I. Institut de Physique, 1 blvd D.Arago, F-57070, Metz, France; e-mail: aberard001@noos.fr.

one-dimensional model in statistical physics. To extrapolate our results in quantum mechanics, adding the one loop contribution due to the reparametrization invariance would be necessary.

In this paper we aim to compare several approaches for the computation of the one-loop wave function renormalization. In order to do so, it is instructive enough to consider a 1D theory in order to avoid the usual UV complications. In particular, we first compute the one loop-renormalization of the kinetic energy with a field dependent wave functions by using a more general expansion than the Frazer's one. This method introduced by Zuk (1985a,b) does not need an expansion about a constant field and leads to different corrections from the ones obtained in Borelli and Kleinert (2000). Our result is a different expression for the effective wave function than the Borelli-Kleinert one.

Two other methods based on the perturbative integration of the “exact” RG equation are also considered. The first one is a RG flow obtained by the mean of a regulator—similar to a smooth cutoff—in the Schwinger Proper Time formalism (Mazzar and Zappala, 2001). Surprisingly it leads to the same result as the one obtained with the method of Zuk. However the weakness of this RG equation is the lack of first principle in its derivation.

The second RG approach is based on another exact equation (ERG) derived in (Gosselin and Mohrbach, 2000; Gosselin and Mohrbach, 2001; Gosselin *et al.*, 2001). It's advantage in comparison to the preceding approach relies on an exact Fourier modes after modes integration in the path integral. This integration is performed without any device like a smooth cutoff. The ERG equation thus obtained as well as the resulting expression for the wave function renormalization are different from the other ones. We provide an explanation for this fact : the perturbative expansion and the smooth cutoff RG method rely on an integration over the all Fourier space, including paths far from the classical one, leading to additional wrong contributions. As a consequence the ERG equation seems to provide the correct one loop wave function renormalization.

## 2. EFFECTIVE ACTION

The effective action formalism in statistical physics is explained in Cametti *et al.* (1999). Here we follow the approach of paper of Borelli and Kleinert (2000).

The starting point is the action in one dimension with a field dependent wave function:

$$\mathcal{A}[\varphi] = \int_{t_a}^{t_b} \left( \frac{Z(\varphi)}{2} \dot{\varphi}^2 - V(\varphi) \right) dt \quad (1)$$

The partition function (Kleinert, 1999) reads in the semi-classical approximation (Borelli and Kleinert, 2000):

$$Z_{QM}(t_b, t_a) [\varphi_{cl}] = e^{\frac{i}{\hbar} \mathcal{A}[\varphi_{cl}]} \int_{\varphi_a = \varphi_b} \mathcal{D}\bar{\varphi} e^{\frac{i}{2\hbar} \int_{t_a}^{t_b} dt \bar{\varphi}(t) K(t) \bar{\varphi}(t)} \tag{2}$$

where  $\bar{\varphi}(t) = \varphi(t) - \varphi_{cl}(t)$  are the fluctuations around the classical path solution of the classical equation of motion, and  $K(t) = \frac{\delta^2 \mathcal{A}}{\delta \varphi(t) \delta \varphi(t)} |_{\varphi_{cl}}$ . The semi-classical effective action is defined as:

$$\mathcal{A}_{eff} [\varphi_{cl}] = -i\hbar \ln Z_{QM} [\varphi_{cl}]$$

which after the gaussian integration in (2) reads:

$$\mathcal{A}_{eff} [\varphi_{cl}] = \mathcal{A}[\varphi_{cl}] - \frac{i\hbar}{2} \text{Tr} \ln K(t) \tag{3}$$

with:

$$K(t) = Z(\varphi_{cl}) \widehat{\omega}^2 - V''(\varphi_{cl}) - iZ'(\varphi_{cl}) \dot{\varphi}_{cl} \widehat{\omega} + \frac{1}{2} Z''(\varphi_{cl}) \dot{\varphi}_{cl}^2 - \frac{d}{dt} (Z'(\varphi_{cl}) \dot{\varphi}_{cl})$$

In Borelli and Kleinert (2000) the one-loop correction was computed by setting  $\varphi_{cl}(t)$  equal to  $\varphi_0 + \tilde{\varphi}(t)$ , where  $\varphi_0$  is constant and expanding the logarithmic term in (3) in powers of  $\tilde{\varphi}(t)$  and its derivatives.

The method of Zuk (1985a,b) we use here, is based on a different kind of expansion in which it is not necessary to expand the classical path  $\varphi_{cl}(t)$  around a constant path. The expansion is obtained by defining  $K(u) = K(t) + u$ , where  $u$  is a (mass)<sup>2</sup> parameter, then by deriving with respect to this parameter and expanding the logarithmic term in (3). In fact by defining  $\Gamma(0) = \frac{i\hbar}{2} \text{Tr} \ln K(t)$  we have the following relation:

$$\Gamma(0) = - \int_0^\infty du \frac{d}{du} \Gamma(u) = -\frac{1}{2} \int_0^\infty du \text{Tr} K^{-1}(u) \tag{4}$$

Now we follow Zuk (1985a,b) by writing the expansion:

$$K^{-1}(u) = A^{-1} - A^{-1} B A^{-1} + A^{-1} B A^{-1} B A^{-1} + \dots$$

where  $A = Z(\varphi_{cl}) w^2 - V''(\varphi_{cl}) - u$  and  $B$  contains all the other contributions of the operator  $K(t)$ . Note that we have introduced the eigenvalue  $w^2$  of the energy operator  $\widehat{w}$  whose coordinate time representation is  $-id/dt$ . Following the same steps as in Zuk (1985a,b) we collect in (4) all the contributions to the coefficient of  $\dot{\varphi}_{cl}^2$  in order to get the effective wave function. A lengthy computation

leads to:

$$Z_{eff}(\varphi_{cl}) = Z(\varphi_{cl}) + \frac{\hbar}{32} \frac{[V'''(\varphi_{cl})]^2 [Z(\varphi_{cl})]^{1/2}}{[V''(\varphi_{cl})]^{5/2}} - \frac{3\hbar}{16} \frac{V'''(\varphi_{cl}) Z'(\varphi_{cl})}{[Z(\varphi_{cl})]^{1/2} [V''(\varphi_{cl})]^{3/2}} \\ - \frac{7\hbar}{32} \frac{[Z'(\varphi_{cl})]^2}{[Z(\varphi_{cl})]^{3/2} [V''(\varphi_{cl})]^{1/2}} + \frac{\hbar}{4} \frac{Z''(\varphi_{cl})}{[Z(\varphi_{cl})]^{1/2} [V''(\varphi_{cl})]^{1/2}} \quad (5)$$

The important and intriguing result is that the various terms in the one-loop expression of  $Z_{eff}(\varphi_{cl})$  have the same form as those obtained in Borelli and Kleinert (2000), but the numerical coefficients are different. For a field-independent wave function we recover the same result found in Borelli and Kleinert (2000) and in Cametti *et al.* (1999).

### 3. ONE LOOP COMPUTATION FROM SCHWINGER PROPER TIME FORMALISM

It is well known that the effective action can be computed nonperturbatively through the Renormalization Group (RG) method. Recently a particular version of the RG flow obtained by means of regulator in the Schwinger Proper Time (PT) formalism has been introduced (Mazzar and Zappala, 2001). This method was used to compute the energy gap between the first excited state and the ground state energy of a one quantum particle system. This work was an alternative to previous computation we performed by means of the exact RG for various quantum systems (Gosselin and Mohrbach, 2000; Gosselin and Mohrbach, 2001; Gosselin *et al.*, 2001).

The flow equation for the running  $Z_k$  in the PTRG formalism is given in Mazzar and Zappala (2001):

$$k \frac{\partial Z_k}{\partial k} = \left( \frac{k^2}{4\pi} \right)^{\frac{1}{2}} e^{-V''/Z_k k^2} \\ \times \left( -\frac{Z''_k}{Z_k k^2} + \frac{21(Z'_k)^2}{24Z_k^2 k^2} + \frac{9Z'_k V'''}{6(Z_k k^2)^2} - \frac{Z_k (V''')^2}{6(Z_k k^2)^3} \right) \quad (6)$$

To compute the one-loop effective wave function we integrate the previous equation from 0 to  $\infty$  keeping the kinetic energy and the potential to their bare values. We directly obtain for the effective wave function:

$$Z_{eff}(\varphi_{cl}) = Z(\varphi_{cl}) + \frac{\hbar}{32} \frac{[V'''(\varphi_{cl})]^2 [Z(\varphi_{cl})]^{1/2}}{[V''(\varphi_{cl})]^{5/2}} - \frac{3\hbar}{16} \frac{V'''(\varphi_{cl}) Z'(\varphi_{cl})}{[Z(\varphi_{cl})]^{1/2} [V''(\varphi_{cl})]^{3/2}} \\ - \frac{7\hbar}{32} \frac{[Z'(\varphi_{cl})]^2}{[Z(\varphi_{cl})]^{3/2} [V''(\varphi_{cl})]^{1/2}} + \frac{\hbar}{4} \frac{Z''(\varphi_{cl})}{[Z(\varphi_{cl})]^{1/2} [V''(\varphi_{cl})]^{1/2}} \quad (7)$$

which is the same result, obtained by the one-loop computation by means of the method of Zuk.

#### 4. ONE LOOP COMPUTATION FROM EXACT RENORMALIZATION GROUP

It is interesting to compare the preceding result with an exact RG equation approach we derived previously (Gosselin and Mohrbach, 2000; Gosselin and Mohrbach, 2001; Gosselin *et al.*, 2001). Working in discrete time allowed us to compute the RG flow one mode after the other in the path integral. The main point is that the RG equation for the effective action is an exact (nonperturbative) equation obtained without any approximation. Note that this is possible only in one dimension since in higher dimension the computation is plagued by non-analytical terms as explained in (Gosselin and Mohrbach, 2000; Gosselin and Mohrbach, 2001; Gosselin *et al.*, 2001) as below. Note that contrary to our equation, the PTRG equation (Mazzar and Zappala, 2001) does not have a first principle derivation.

The running wave function equation is (Gosselin and Mohrbach, 2000; Gosselin and Mohrbach, 2001; Gosselin *et al.*, 2001):

$$Z_{n-1}(\varphi_0) = Z_n(\varphi_0) + \frac{1}{2\beta} \left( 1 + \frac{Z_n''(\varphi_0)}{Z\omega_n^2 + V''} \right) \tag{8}$$

where  $n$  denotes the  $n$ th discrete mode and  $\omega_n$  is the corresponding Fourier mode. Remember that  $\beta = \frac{(N+1)\epsilon}{\hbar}$  is the inverse temperature (Kleinert, 1999) and  $\epsilon$  is the time interval (lattice spacing). In the zero temperature limit  $N\epsilon \rightarrow \infty$ , and the continuum limit  $\epsilon \rightarrow 0$  Eq. (8) becomes:

$$\frac{\partial Z_k(\varphi_0)}{\partial k} = \frac{\hbar}{2\pi} \frac{Z_k''(\varphi_0)}{Z_k(\varphi_0)k^2 + V_k''(\varphi_0)} \tag{9}$$

where  $k$  is the energy cut-off. By solving this equation at the one-loop level we are led to:

$$Z_{eff}(\varphi_0) = Z(\varphi_0) + \frac{\hbar}{4} \frac{Z''(\varphi_0)}{[Z(\varphi_0)]^{1/2} [V''(\varphi_0)]^{1/2}}$$

This expression is in contradiction with those obtained by Kleinert and with the method of Zuk. The only common contribution is the last term in (5). Note that as a consequence of (9) there is no renormalization effect for a constant bare wave function:  $Z_{eff}(\varphi_0) = Z(\varphi_0)$ . This result agrees with some numerical computations done in Jirari *et al.* (2002).

The absence of the other terms in (9) has been already explained in (Gosselin and Mohrbach, 2000; Gosselin and Mohrbach, 2001; Gosselin *et al.*, 2001) and is related to the presence of the so-called non analytical terms when the Fourier modes are continuous. In field theory these terms appear when the integration on

the fast modes is performed on a continuous spherical shell of thickness  $\Delta k$ . It is precisely these terms which give the additional contribution to the flow of the kinetic energy if we forget the non-analytical terms.

To precise our argument let us consider the derivation of the RG flow for the wave function by integrating the Fourier modes in a continuous shell of thickness  $\Delta k$ . For the point we want to show it is enough to consider a space independent wave function. We arrive at the following equation (Gosselin and Mohrbach, 2000; Gosselin and Mohrbach, 2001; Gosselin *et al.*, 2001):

$$Z_{k-\Delta k} q^2 + U_{k-\Delta k}^{(2)}(\varphi_0) = Z_k q^2 + U_k^{(2)}(\varphi_0) + U_k^{(4)}(\varphi_0) \int_k^{k-\Delta k} \frac{dp}{2\pi} \frac{1}{G(p)} + F(q)$$

where

$$F(q) = \frac{(U_k^{(3)}(\varphi_0))^2}{4} \int_k^{k-\Delta k} \frac{dp}{2\pi} \int_k^{k-\Delta k} \frac{dp'}{2\pi} \frac{\delta(p+p'+q) + \delta(p+p'-q)}{G(p)G(p')} + \text{H.c.} \quad (10)$$

It is clear that for  $q < \Delta k$ , this integral gives a contribution of the form  $\Delta k - |q|$  because the domain of integration is deformed by the Dirac delta constraints. By expanding the denominator in powers of  $q$ , we obtain the non-analytical contribution.:

$$F(q) = \frac{(U_k^{(3)}(\varphi_0))^2}{4} \frac{\Delta k - |q|}{G^2(k)} (1 + O(q^2)) \quad \text{for } q < \Delta k$$

and

$$F(q) = 0 \quad \text{for } q \geq \Delta k$$

Retrieving the first order quantum correction to the classical equation derived firstly by Jona-Lasinio's group, as well as Borelli-Kleinert result, corresponds to neglect the  $|q|$  contribution and let

$$F(q) = \frac{(U_k^{(3)}(\varphi_0))^2}{4} \frac{\Delta k}{G^2(k)}$$

However, it is obvious that a rigorous computation of the discrete version of Eq. (10) leads to the result  $F(q) = 0$ , yielding formula (9) in the limit  $\Delta k \rightarrow 0$ .

The cancellation of the  $F(q)$  contribution shows an important difference between the Renormalization Group and the perturbative expansion. In the last one, the semi classical contributions are obtained by integrating over all Fourier space, after Taylor expanding the action around a classical path. This is problematic, since it corresponds to integrate over all paths including those far from the classical path. On the contrary, the Renormalization Group approach, by introducing integration on deformed shells of possible zero measure (see Gosselin and Mohrbach, 2000; Gosselin and Mohrbach, 2001; Gosselin *et al.*, 2001), leads to the cancellation of

some terms that are relevant in the perturbative or semi classical approach, like the term  $F(q)$ . This cancellation is simply reminiscent of the difference between integrating a two variables function over a square  $[0, \Lambda]^2$  (as in (10), where the two variables are  $p$  and  $p'$ ) and the integration over small intervals  $[k, k + \Delta k]^2$ ,  $k \in [0, \Lambda]$  and then summing over the squares. In the first case the integral is performed over a bigger space. Note that this problem only arises for the renormalization of the wave function and is absent for the computation of the corrections to the potential energy. Actually in this last case, the perturbative expansion is based on an integration over  $[0, \Lambda]$  whereas the perturbative integration of the RG equation introduces a sum of integration over the one dimensional intervals  $[k, k + \Delta k]$ ,  $k \in [0, \Lambda]$ . The two procedures are obviously equivalent in that case.

In that context we can now explain why the perturbative integration of the PTRG equation leads to the same result as the perturbative method (in particular the result of Zuk). Indeed, in the derivation of the PTRG equation one has to replace the double integration of the Dirac delta in (10) by only one integral with a smooth cutoff. This unique integral is performed over all the modes, like in the perturbation computation, and this leads to a non-zero contribution of  $F(q)$ . The smooth cutoff device, while avoiding the problem of the non-analytical terms, relies on integration over a too large Fourier space leading to take into account wrong additional terms.

The difference between the usual semi classical approximation and the Renormalization Group makes us believe that the correct one-loop result is retrieved after integrating the Exact Renormalization Group differential equation. As a consequence, the perturbative integration of the RG equation we have proposed (9) seems to be the correct one-loop wave function renormalization, but contradicts the traditional perturbative methods.

## 5. CONCLUSION

We have computed the one-loop quantum correction of the kinetic energy for a field dependent wave function in one dimension in order to compare various methods. The main result of the paper is that the perturbative integration of the ERG equation leads to the exact one-loop wave function renormalization. This result contradicts the traditional perturbative methods.

## REFERENCES

- Borelli, M. E. S. and Kleinert, H. (2000). quant-ph/0006135.
- Cametti, F. *et al.* (1999). quant-ph/9910065.
- Fraser, C. M., (1985). *Zeitschrift für Physikalische Chemie* **28**, 101.
- Gosselin, P. and Mohrbach, H. (2000). *Journal of Physics A: Mathematical and General* **33**, 6343.
- Gosselin, P. and Mohrbach, H. (2001). *Physics Letters A* **284**, 6.

- Gosselin, P., Mohrbach, H., and Bérard, A. (2001). *Physics Review E* **64**, 046129.
- Iliopoulos J. *et al.* (1975). *Reviews of Modern Physics* **47**, 165.
- Jirari, H. *et al.* (2002). *Physics Letters A* **303**, 299.
- Kleinert, H. (1999). *Path Integrals in quantum Mechanics Statistics and Polymer Physics*, World Scientific, 2nd ed.
- Kleinert, H. and Chervyakov, H. (2002). *Physics Letters A* **299**, 319.
- Mazzar, M. and Zappala, D. (2001). *Physics Review D* **64**, 105013.
- Zuk, J. A. (1985a). *Physics Review D* **32**, 2653.
- Zuk, J. A. (1985b). *Journal of Physics A: Mathematical and General* **18**, 1795.